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1975 J. Phys. A: Math. Gen. 8 76

(http://iopscience.iop.org/0305-4470/8/1/014)

J. Phys. A: Math. Gen., Vol. 8, No. 1, 1975. Printed in Great Britain.

A composition law for solutions of Weisskopf–Wigner theories

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Received 12 June 1974, in final form 19 August 1974

Abstract. We treat the interaction of incident photons with an atom in a Weisskopf–Wigner theory on an arbitrarily large Hilbert space defined by some arbitrarily weak important-state hypothesis. It is shown rigorously that the solution describing the interaction of arbitrarily many incident photons in an arbitrary initial state of coherence or incoherence with an atom in any initial state can be 'composed' of simpler elements, namely: amplitudes of one free photon, amplitudes of spontaneously emitted photons, and amplitudes of photons emitted in a resonance fluorescence process after the absorption of one, two, ... incident photons. The composition law is applied to some problems in the interaction of a single atom with a coherent beam of incident photons.

1. Introduction

Grimm and Ernst (1974, to be referred to as GE) have recently systematized an approximation idea of Weisskopf and Wigner (1930) (see also Dirac 1927a, Weisskopf 1931, Källén 1958, Ernst and Stehle 1968, and others) to a well structured hierarchy of 'Weisskopf–Wigner (WW) approximations' or 'WW theories' for treating the interaction of photons with an atom. Whereas GE mainly analysed the existence of WW theories we look here at the practical computation of solutions. We show that, and how, the most general solution describing the interaction of a bound electron with incident photons can be 'composed' of the simpler amplitudes noted in the abstract. These constituents are either known or defined as special solutions of the given WW theory or of simpler WW theories involving fewer, in cases of practical interest frequently very much fewer, photons. Their physical content, in addition, is quite obvious in many cases. We show that they exist at least in all those cases where the considered WW theory exists with certainty.

Our composition law, formally a 'symmetrization procedure', lays open the *structure* of the desired solution. In problems involving many photons (optical coherence, for example) this is frequently more important than the detailed knowledge of the constituents. It can be applied directly to many photons in any pure state impinging upon one atom, but generalizations to cases with many atoms and/or incomplete knowledge of the initial state appear straightforward. Specializations, eg to a fully coherent initial state, are simple applications. The most striking feature of the composition law is its existence at all, and that even in any order of the systematic WW approximation scheme of GE. The possibility of composing any solution of practically any WW theory of constituents obtained from WW theories involving fewer photons demonstrates quite

obviously that the GE hierarchy of WW approximations follows 'natural lines' of the exact theory. It thus exposes exact structures in the atom–photon interaction.

In view of the complexity of the composition law we concentrate our attention on its proper formulation and proof. The law seems to be without direct precedents in the literature; in particular, we find no comparable ideas in previous WW theories with incident photons (Weisskopf 1931, Bergmann 1967). Some of the ideas in the proof go back to the work of Ernst and Stehle (1968). To illustrate its potential and its working mechanism we apply it to some simple problems in the interaction of an atom with a coherent beam of incident photons.

An outline of the GE concept is given in § 2. In § 3 we formulate the composition law. The existence and structure of the constituents needed are analysed in § 4. The proof is completed in § 5. Sections 6–8 contain the applications mentioned.

2. A review of the systematic Weisskopf–Wigner approximation idea

We introduce the notation and background material by giving an outline of the WW approximation scheme of GE.

As in GE we consider a slightly modified one-electron Dirac atom A with a discrete energy spectrum E^a and eigenstates $u_a(x)$, the index a comprising all necessary atomic quantum numbers. By the transverse part $eA \cdot J$ of the usual minimal coupling density $eA_\mu J^\mu$ the atom A interacts with the 'radiation field' R of transverse photons. To control the infrared behaviour we equip them with a formal mass $\mu \geqslant 0$. GE define a WW approximation of this problem as a Schrödinger theory on the Hilbert space

$$\mathscr{H}(I) = \bigoplus_{(a,n)\in I} \mathscr{H}_n^a \tag{1}$$

where $\oplus$ denotes the 'orthogonal sum' and $\mathscr{H}_n^a$ is the Hilbert space of all states of the coupled system $A+R$ with the property 'A in the state $u_a(x)$ *and* R in some state with $n \geqslant 0$ photons'. Let $\mathscr{K} := \mathbb{R}^3 \times \{1, 2\}$ denote the momentum and polarization space of a photon with elements $\kappa := (\boldsymbol{k}, \lambda)$ comprising wavevector $\boldsymbol{k}$ and polarization index λ. $\mathscr{H}_n^a$ then is the subspace of, in $\kappa_1, \ldots, \kappa_n$, symmetric elements $\alpha_n^a := \alpha_n^a(\kappa_1, \ldots, \kappa_n)$ of $L^2(\mathscr{K}^{\times n})$, with $\mathscr{K}^{\times n} := \mathscr{K} \times \ldots \times \mathscr{K}$ (n times) and $L^2(\mathscr{K}^{\times n}) := \mathbb{C}$ for $n = 0$. I is some finite or infinite set of index pairs (a, n) which is determined by some 'important state hypothesis' and defines the 'order' of the resulting WW approximation. With $\omega(\kappa) = \omega(\boldsymbol{k}) := (\boldsymbol{k}^2 + \mu^2)^{1/2}$ ($\hbar = c = 1$) and $M(a, b; \kappa)$ defined below, in terms of the '(a, n) components' $\alpha(t)_n^a(\kappa_1, \ldots, \kappa_n)$ of a vector $|\alpha_t(t)\rangle \in \mathscr{H}(I)$ the Schrödinger theory on $\mathscr{H}(I)$ reads

$$\begin{aligned}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)_n^a(\kappa_1, \ldots, \kappa_n) &= (\omega(\kappa_1) + \ldots + \omega(\kappa_n) + E^a)\alpha(t)_n^a(\kappa_1, \ldots, \kappa_n) \\
&\quad + \sum_{(b,n+1)\in I} (n+1)^{1/2} \int \mathrm{d}^3\kappa M^*(a, b; \kappa)\alpha(t)_{n+1}^b(\kappa, \kappa_1, \ldots, \kappa_n) \\
&\quad + \sum_{(b,n-1)\in I} \frac{1}{n^{1/2}} \sum_{\mu=1}^{n} M(b, a; \kappa_\mu)\alpha(t)_{n-1}^b(\kappa_1, \ldots, \kappa_{\mu-1}, \kappa_{\mu+1}, \ldots, k_n).
\end{aligned} \tag{2}$$

$\int \mathrm{d}^3\kappa \ldots$ denotes integration over $\boldsymbol{k}$ *and* summation over λ. Equation (2) holds simultaneously for all elements (a, n) of I and the sums on the right-hand side extend over all

pairs $(b, n\pm 1)$ which, together with the element (a, n) given on the left-hand side, belong to I. The initial conditions are

$$\alpha(0)^a_n(\kappa_1, \ldots, \kappa_n) = \chi^a_n(\kappa_1, \ldots, \kappa_n) \tag{3}$$

with χ^a_n denoting the (a, n) components of any vector $|\chi_I\rangle \in \mathscr{H}(I)$.

The existence of a WW theory on $\mathscr{H}(I)$ of course depends on the 'atomic transition elements'

$$M^*(a, b; \kappa) := \frac{e\epsilon(\kappa)}{[2\omega(\kappa)(2\pi)^3]^{1/2}} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}k\cdot x} u^\dagger_a(\boldsymbol{x}) \boldsymbol{\alpha} u_b(\boldsymbol{x}). \tag{4}$$

$\int \mathrm{d}^3x \ldots$ denotes the usual integration over $\mathbb{R}^3$, e is the coupling constant, $\boldsymbol{\alpha}$ the usual Dirac vector, and $\epsilon(\kappa)$ the usual unit transverse polarization vector. GE showed that $M(a, b, \kappa)$ is an element of $L^2(\mathscr{K})$ for any two bound states $u_a(\boldsymbol{x})$, $u_b(\boldsymbol{x})$ of the ordinary Dirac hydrogen atom. They showed further that the right-hand side of (2) defines on $\mathscr{H}(I)$ a self-adjoint Hamilton operator H_I if

$$M(a, b; \kappa) \in L^2(\mathscr{K}) \qquad \text{for any occurring } a, b \tag{5}$$

and if, in addition, either

$$\mu \geqslant 0, \qquad I \text{ finite} \tag{6a}$$

or

$$\mu > 0, \qquad I = \{Q \times \mathbb{N}_0\}. \tag{6b}$$

Q denotes any *finite* set of atomic indices a and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. The rigorous self-adjointness of H_I on $\mathscr{H}(I)$ guarantees the existence of a unique, unitary time evolution operator $U_I(t)$ on $\mathscr{H}(I)$ for all cases (6). Condition (5) is vital for this existence and will be assumed to hold throughout this work.

If $I_0 \subset I_1 \subset \ldots$ is any sequence of index sets which tends to the set $I_{\mathscr{S}}$ of all possible pairs (a, n) one hopes that the sequence of WW approximations on the sequence of Hilbert spaces $\mathscr{H}(I_0) \subset \mathscr{H}(I_1) \subset \ldots$ 'tending' to the exact state space $\mathscr{S}$ of $A+R$ in some sense might converge against some 'exact' theory on $\mathscr{S}$. Not much is known about this convergence, however. Källén's (1958) version of the original theory of Weisskopf and Wigner (1930) is the simplest example of a WW theory with a two-element set I obeying (6a). The success of such simple WW theories (proper linewidths in the original theory of Weisskopf and Wigner (1930), expected linewidths and directivity effects in Weisskopf's (1931) theory of one-photon resonance fluorescence, directivity (Ernst and Stehle 1968) and coherence (Ernst 1969) in modern super-radiance problems) encourages work on more elaborate WW theories, like that to be presented now.

3. The composition law

We consider now the WW theory on an arbitrary $\mathscr{H}(I)$ and formulate the composition law.

Let $b_1(\kappa), b_2(\kappa), \ldots$ be the elements of a complete, orthonormal base in $L^2(\mathscr{K})$. For convenience we restrict its choice by the requirement that all $b_i(\kappa)$ shall be in the intersection of the domains of the multiplication operators $\omega(\kappa)$ and $(\omega(\kappa))^2$ so that the integrals

$$g^{a,c}_i(t) := \int \mathrm{d}^3\kappa M^*(a, c; \kappa) b_i(\kappa)\, \mathrm{e}^{-\mathrm{i}\omega(\kappa)t} \tag{7}$$

because of (5) define uniformly bounded, continuously differentiable functions of t. Such bases exist; for given λ the $b_i(\boldsymbol{k}, \lambda)$ could be the Fourier transforms of the eigenfunctions of the harmonic oscillator in three dimensions, for example.

Further, let $i_1, \ldots, i_n$ be any choice of natural numbers and let $[i_1, \ldots, i_n]$ be the class of those choices $i_1, \ldots, i_n$ which can be obtained from each other by a permutation of the elements. To any class $[i_1, \ldots, i_n]$ we define its 'permutability'

$$p[i_1, \ldots, i_n] = \frac{n!}{n_1! n_2! \ldots} \tag{8}$$

where n_s is the number of times the natural number s occurs among the numbers i of any choice $i_1, \ldots, i_n$. To any class we associate an element

$$\varphi[i_1, \ldots, i_n]_n^a(\kappa_1, \ldots, \kappa_n)$$

$$= (p[i_1, \ldots, i_n])^{1/2} \frac{1}{n!} \mathop{\mathsf{P}}_{j_1, \ldots, j_n}^{i_1, \ldots, i_n} b_{j_1}(\kappa_1) \ldots b_{j_n}(\kappa_n)$$

$$= (p[i_1, \ldots, i_n])^{1/2} \frac{1}{n!} \mathop{\mathsf{P}}_{\eta_1, \ldots, \eta_n}^{\kappa_1, \ldots, \kappa_n} b_{i_1}(\eta_1) \ldots b_{i_n}(\eta_n) \tag{9}$$

of $\mathscr{H}_n^a$ where

$$\mathop{\mathsf{P}}_{x_1, \ldots, x_n}^{y_1, \ldots, y_n} f(x_1, \ldots, x_n) := f(y_1, \ldots, y_n) + f(y_2, y_1, y_3, \ldots, y_n) + \ldots \tag{10}$$

denotes the sum of the $n!$ terms obtained by inserting into $f(x_1, \ldots, x_n)$ all $n!$ formal permutations of the elements $y_1, \ldots, y_n$. This 'permutation sum' (Ernst and Stehle 1968) is a symmetric function of $y_1, \ldots, y_n$ which reduces to a factor $n!$ if f is symmetric already. Further, for $n = 0$, let the above quantities be defined as $p[\phi] = 1$, $\varphi_0^a = 1$ ($\in \mathbb{C}$), and the permutation sum as a factor $1 \cdot \varphi_0^a$ or, for $n > 0$, the set of all vectors (9), are a complete, orthonormal base in $\mathscr{H}_n^a$. Therefore, any initial component $\chi_n^a(\kappa_1, \ldots, \kappa_n) \in \mathscr{H}_n^a$ can be expanded in the form

$$\chi_n^a(\kappa_1, \ldots, \kappa_n) = \sum_{i_1, \ldots, i_n} \frac{(p[i_1, \ldots, i_n])^{1/2} \chi_n'^a(i_1, \ldots, i_n)}{p[i_1, \ldots, i_n]} \varphi[i_1, \ldots, i_n]_n^a(\kappa_1, \ldots, \kappa_n) \tag{11}$$

with the multiple sum

$$\sum_{i_1, \ldots, i_n} \ldots := \sum_{i_1=1}^{\infty} \ldots \sum_{i_n=1}^{\infty} \ldots \tag{12}$$

in the case $n = 0$ defined as factor 1. The factor p^{-1} in (11) takes care of the fact that the sum (12) counts each element of the class $[i_1, \ldots, i_n]$ precisely once and thus overcounts the class by a factor $p[i_1, \ldots, i_n]$. The factor $p^{1/2}$ in (11) compensates a factor $p^{-1/2}$ which has been defined into the expansion coefficient $\chi_n'^a$ for convenience. The simplicity so obtained has deep lying reasons which need not be discussed here. Inserting (9) into (11) we get

$$\chi_n^a(\kappa_1, \ldots, \kappa_n) = \sum_{i_1, \ldots, i_n} \chi_n'^a(i_1, \ldots, i_n) b_{i_1}(\kappa_1) \ldots b_{i_n}(\kappa_n). \tag{13}$$

Here we have exploited the fact that under the sum (12) the first permutation sum in (9) yields merely a factor $n!$. The $\chi'^a_n(i_1, \ldots, i_n)$ are explicitly given by

$$\chi'^a_n(i_1, \ldots, i_n) = \int \mathrm{d}^3\kappa_1 \ldots \int \mathrm{d}^3\kappa_n b^*_{i_1}(\kappa_1) \ldots b^*_{i_n}(\kappa_n)\chi^a_n(\kappa_1, \ldots, \kappa_n) \tag{14}$$

($\chi'^a_0 = \chi^a_0$) and satisfy the normalization condition

$$1 = \sum_{(a,n)\in I} \int \mathrm{d}^3\kappa_1 \ldots \int \mathrm{d}^3\kappa_n |\chi^a_n(k_1, \ldots, k_n)|^2 = \sum_{(a,n)\in I} \sum_{i_1,\ldots,i_n} |\chi'^a_n(i_1, \ldots, i_n)|^2. \tag{15}$$

In the case $n = 0$ the multiple sum and multiple integral shall be understood as a factor 1 so that the terms $|\chi'^a_0|^2 = |\chi^a_0|^2$ are included properly.

After these necessary preparations we are ready to formulate the composition law. *Part A.* For the WW theory defined by equations (2) on any $\mathscr{H}(I)$ the expression

$$\begin{aligned}\alpha(t)^a_n(\kappa_1, \ldots, \kappa_n):&\\ &= \sum_{(b,m)\in I} \sum_{i_1,\ldots,i_m} \chi'^b_m(i_1, \ldots, i_m) \left(\frac{m!}{n!}\right)^{1/2} \sum_{\rho=0}^{\min(m,n)} \frac{1}{\rho![(n-\rho)!]^{1/2}}\\ &\quad \times \mathop{\mathbf{P}}_{\eta_1,\ldots,\eta_n}^{\kappa_1,\ldots,\kappa_n} b_{i_1}(\eta_1)\,\mathrm{e}^{-\mathrm{i}\omega(\eta_1)t} \ldots b_{i_\rho}(\eta_\rho)\,\mathrm{e}^{-\mathrm{i}\omega(\eta_\rho)t}\\ &\quad \times \beta(^{b;n}_{m-\rho};i_{\rho+1}, \ldots, i_m; t)^a_{n-\rho}(\eta_{\rho+1}, \ldots, \eta_n)\end{aligned} \tag{16}$$

exists for any $(a, n) \in I$ and is the (a, n) component of the vector $U_I(t)|\chi_I\rangle$ which in accordance with equations (2) evolves in time from the arbitrary initial vector $|\chi_I\rangle \in \mathscr{H}(I)$ with components $\chi^a_n(\kappa_1, \ldots, \kappa_n)$, provided the following three conditions are satisfied.

(i) Equations (2) define on $\mathscr{H}(I)$ for any $t < \infty$ a unique and unitary time evolution operator $U_I(t)$.

(ii) Let (a, n), (b, m) be any, not necessarily different elements of I, let ν, σ be integers of the sets $\nu \in \{0, \ldots, n\}$ and $\sigma \in \{0, \ldots, m\}$, and finally to any given $\sigma \geqslant 1$ let $(j_1, \ldots, j_\sigma)$ be any set of σ natural numbers. To any choice (a, n), (b, m), ν, σ, $j_1, \ldots, j_\sigma$ there exists a function $\beta(^{b;n}_{\sigma}; j_1, \ldots, j_\sigma; t)^a_\nu(\kappa_1, \ldots, \kappa_\nu)$ ($\equiv 0$ for $\sigma < 0$) of t and of $\kappa_1, \ldots, \kappa_\nu$ which for $\nu \geqslant 1$ is symmetric in $\kappa_1, \ldots, \kappa_\nu$ and defined on $\mathscr{K}^{\times\nu}$. Together with the functions corresponding to other choices (a', n'), (b', m'), ν', σ', $j'_1, \ldots, j'_\sigma$ it satisfies the equations of motion

$$\begin{aligned}&\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\beta(^{b;n}_{\sigma}; j_1, \ldots, j_\sigma; t)^a_\nu(\kappa_1, \ldots, \kappa_\nu)\\ &= (\omega(\kappa_1) + \ldots + \omega(\kappa_\nu) + E^a)\beta(^{b;n}_{\sigma}; j_1, \ldots, j_\sigma; t)^a_\nu(\kappa_1, \ldots, \kappa_\nu)\\ &+ (\nu+1)^{1/2} \sum_{(c,n+1)\in I} \int \mathrm{d}^3\kappa M^*(a, c; \kappa)\beta(^{b;n+1}_{\sigma}; j_1, \ldots, j_\sigma; t)^c_{\nu+1}(\kappa, \kappa_1, \ldots, \kappa_\nu)\\ &+ \frac{1}{\nu^{1/2}} \sum_{(c,n-1)\in I} \sum_{\mu=1}^{\nu} M(c, a; \kappa_\mu)\beta(^{b;n-1}_{\sigma}; j_1, \ldots, j_\sigma : t)^c_{\nu-1}(\kappa_1, \ldots, \kappa_{\mu-1}, \kappa_{\mu+1}, \ldots, \kappa_\nu)\\ &+ \sum_{(c,n+1)\in I} g^{a,c}_{j\sigma}(t)\,\beta(^{b;n+1}_{\sigma-1}; j_1, \ldots, j_{\sigma-1}; t)^c_\nu(\kappa_1, \ldots, \kappa_\nu)\end{aligned} \tag{17}$$

under initial conditions

$$\beta(^{b;n}_{\sigma};\, j_1, \ldots, j_\sigma;\, 0)^a_\nu(\kappa_1, \ldots, \kappa_\nu) = \delta_{b,a}\delta_{\sigma,0}\delta_{\nu,0} \qquad \text{for any } n. \tag{18}$$

(iii) The sums over $(b, m) \in I$ and $i_1, \ldots, i_m$ in (16) converge so that the operator H_I defined by (2) may be applied under these sums.

Part B. Conditions (i), (ii) and (iii) of part A are satisfied at least for all WW theories satisfying conditions (5) and (6).

We have no intention here to extend the existence conditions for WW theories and so condition (i) has been chosen partly for convenience. Condition (iii) is extremely weak and of technical relevance only, if at all. For $|\chi_I\rangle$ from the domain of H_I the equation (16) of course yields the solution of (2). Since $U_I(t)$ is unique, also the closed expression (16) is unique though its single constituents, the $b_i(\kappa)$ and the β functions, of course depend on the choice of the base (9). Part A is formulated so that a *formal* composition can be made in any WW theory. Part B puts all this on a rigorous base where such a base exists with certainty.

It is important to note that in (16) the sums over (b, m) and $i_1, \ldots, i_m$ in essence cover only the contributions of the vectors of a complete base in $\mathscr{H}(I)$ so that these sums are a trivial consequence of the quantum-mechanical superposition principle. The point of the theorem is the 'composition' of each of these contributions. We see that the contribution of the m-photon component of any initial state to the solution never contains more than m 'free-photon factors' $b_i(\kappa)\, \mathrm{e}^{-\mathrm{i}\omega(k)t}$: the incident photons are replaced, step by step, by β functions. This is the decisive point and the key to the physical interpretation. The composition law becomes considerably simpler in certain special cases, for example, if the initial state is fully coherent. We look at this in § 6.

4. Structure and existence of the constituents

We prepare the proof of the composition law with an investigation of the constituents defined by (17) and (18).

Looking at (17) we first note a recursion with respect to σ: if the $\beta(^{\cdots}_{\sigma-1}\ldots)$ are given, the last term in (17) is known and we have to solve an inhomogeneous linear system of equations for the $\beta(^{\cdots}_{\sigma}\ldots)$. We call the set G_σ of all β's with the same σ a 'generation'. The 'founder generation' G_0 owes its existence to the non-vanishing initial values (18), all 'daughter generations' G_σ, $\sigma \geqslant 1$, have vanishing initial values and thus depend on the existence of preceding 'parent generations' $G_{\sigma-1}$ which, by the last term of (18), become their source.

Considering equations (17) for some fixed value of σ we see further that, with the last term given, only $\beta(^{\cdots}_{\sigma}\ldots)$'s with constant difference τ between n and ν ($\tau = n - \nu$) are *connected* by the equations of motion. We call the set $G^\tau_\sigma \subseteq G_\sigma$ of all elements $\beta(^{\cdots}_{\sigma}\ldots)$ corresponding to the same value of τ a 'family' so that each generation G_σ is the union of its families G^τ_σ. We note that each β occurring in (16) belongs to precisely one family G^τ_σ and that only the families with $\tau \geqslant 0$ occur in the theory.

We first look at G_0 and write down the equations of motion for some founder family G^τ_0. With $n = \nu + \tau$ we obtain for

$$\beta(^{b;\tau+\nu}_{0};\, \phi;\, t)^a_\nu(\kappa_1, \cdots, \kappa_\nu) =: \beta(^{b;\tau+\nu}_{0};\, t)^a_\nu(\kappa_1, \ldots, \kappa_\nu)$$

the equations

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\beta(^{b;\tau+\nu}_{0};t)^{a}_{\nu}(\kappa_1,\ldots,\kappa_\nu)$$

$$=(\omega(\kappa_1)+\ldots+\omega(\kappa_\nu)+E^{a})\beta(^{b;\tau+\nu}_{0};t)^{a}_{\nu}(\kappa_1,\ldots,\kappa_\nu)$$

$$+(\nu+1)^{1/2}\sum_{(c,\tau+\nu+1)\in I}\int \mathrm{d}^3\kappa M^{*}(a,c;\kappa)\beta(^{b;\tau+\nu+1}_{0};t)^{c}_{\nu+1}(\kappa,\kappa_1,\ldots,\kappa_\nu)$$

$$+\frac{1}{\nu^{1/2}}\sum_{(c,\tau+\nu-1)\in I}\sum_{\mu=1}^{\nu}M(c,a;\kappa_\mu)\beta(^{b;\tau+\nu-1}_{0};t)^{c}_{\nu-1}$$

$$\times(\kappa_1,\ldots,\kappa_{\mu-1},\kappa_{\mu+1},\ldots,\kappa_\nu) \tag{19}$$

with initial conditions

$$\beta(^{b;\tau+\nu}_{0};0)^{a}_{\nu}(\kappa_1,\ldots,\kappa_\nu)=\delta_{a,b}\delta_{\nu,0}\qquad\text{for any }\tau. \tag{20}$$

In these equations b and τ are fixed parameters and $(a,\nu),(c,\nu\pm1)$ are restricted by the condition that $(a,\tau+\nu),(c,\tau+\nu\pm1)$ are elements of I. They are therefore elements of the set

$$I^{\tau}:=\{(a,\nu):\nu\geqslant 0,(a,\tau+\nu)\in I\},\qquad \tau=0,1,2,\ldots \tag{21}$$

which obviously never contains more elements than I. It is suggestive to consider (19) as an equation on the Hilbert space $\mathscr{H}(I^{\tau})$. For comparison, let us consider the WW theory (2) on this Hilbert space: equations (2) have to be written down for any (a,ν) with the properties $\nu\geqslant 0$ and $(a,\tau+\nu)\in I$, as also requested for (19). Further, on the right-hand side of (2) we have to sum over all elements $(c,\nu\pm1)\in I^{\tau}$, ie for given (a,ν) with $\nu\geqslant 0$ and $(a,\tau+\nu)\in I$ over all elements $(c,\nu\pm1)$ with $\nu\pm1\geqslant 0$ and $(c,\tau+\nu\pm1)\in I$. But these are precisely the summation requirements of (19). Finally, the same $M(a,c;\kappa)$ as in (19) occur in the WW theory on $\mathscr{H}(I^{\tau})$. It follows that equations (19) are identical with the WW theory on $\mathscr{H}(I^{\tau})$. Since to an element (a,n) of I 'corresponds' an element $(a,n-\tau)$ of I^{τ}, the theory on $\mathscr{H}(I^{\tau})$ is obtained by 'shifting the given theory on $\mathscr{H}(I)$ to fewer photons'. The members of a family G^{τ}_{0} thus are the components of a solution vector $|\alpha_{I^{\tau}}(t)\rangle$ of a WW theory of fewer photons. If I is finite, all I^{τ} are also finite. Since the theory on $\mathscr{H}(I^{\tau})$ involves no $M(a,c;\kappa)$ other than the theory on $\mathscr{H}(I)$, it exists whenever this theory exists. So for (6*a*) as in GE we get a unique, norm-conserving solution of (19), (20). In the case (6*b*) we obviously have $I^{\tau}=I$ for any τ so that all families obey the same equations of motion. The upper index n of the β functions is superfluous here and can be omitted. All members exist and are unique, as in GE. Since the initial states in both cases are from the domain of $H_{I^{\tau}}$ we had the right to speak of 'solutions' directly.

These solutions are continuously differentiable with respect to t. For the above reason the right-hand side of (19), now considered with the given special solution $\beta(^{\cdot\cdot}_{0}\ldots)$ inserted, exists, and with it the left-hand side, ie the time derivatives of the β's. But if the inserted β's are differentiable, the right-hand side of (19) is continuous in practically any case of interest, and then also the time derivatives of the β's are continuous.

We note finally that because of (20) only those founder families have non-vanishing members whose I^{τ} contains the element $(b,0)$. For proper τ such families exist. The families G^{m}_{0} contribute for $m\leqslant n$ the terms $\rho=m$ of (16). Since (b,m) is from I, $(b,0)$ is in I^{m} and so the G^{m}_{0} have non-trivial elements. This guarantees that (16) can assume the required initial values.

We now proceed to the first daughter generation G_1 and pick out some family G_1^τ. Its equations of motion are

$$\begin{aligned}
i\frac{d}{dt}\beta(^{b;\tau+\nu}_{1};j_1;t)^a_\nu(\kappa_1,\ldots,\kappa_\nu)
&\times(\omega(\kappa_1)+\ldots+\omega(\kappa_\nu)+E^a)\beta(^{b;\tau+\nu}_{1};j_1;t)^a_\nu(\kappa_1,\ldots,\kappa_\nu)\\
&+(\nu+1)^{1/2}\sum_{(c,\tau+\nu+1)\in I}\int d^3\kappa M^*(a,c;\kappa)\beta(^{b;\tau+\nu+1}_{1};j_1;t)^c_{\nu+1}(\kappa,\kappa_1,\ldots,\kappa_\nu)\\
&+\frac{1}{\nu^{1/2}}\sum_{(c,\tau+\nu-1)\in I}\sum_{\mu=1}^{\nu}M(c,a;\kappa_\mu)(^{b;\tau+\nu-1}_{1};j_1;t)^c_{\nu-1}\\
&\times(\kappa_1,\ldots,\kappa_{\mu-1},\kappa_{\mu+1},\ldots,\kappa_\nu)+q_1^\tau(t)^a_\nu(\kappa_1,\ldots,\kappa_\nu)
\end{aligned}\tag{22}$$

with initial conditions

$$\beta(^{b;\tau+\nu}_{1};j_1;0)^a_\nu(\kappa_1,\ldots,\kappa_\nu)=0\qquad\text{for all } a,\nu,\tau.\tag{23}$$

The last term of (22) is

$$q_1^\tau(t)^a_\nu(\kappa_1,\ldots,\kappa_\nu):=\sum_{(c,\tau+\nu+1)\in I}g^{a,c}_{j_1}(t)\beta(^{b;\tau+1+\nu}_{0};t)^c_\nu(\kappa_1,\ldots,\kappa_\nu).\tag{24}$$

The summation conventions are the same as in (19) so that (22) can be looked upon as an inhomogeneous linear equation on $\mathscr{H}(I^\tau)$,

$$i\frac{d}{dt}|\beta_{I^\tau}(t)\rangle=H_{I^\tau}|\beta_{I^\tau}(t)\rangle+|q_1^\tau(t)\rangle,\qquad|\beta_{I^\tau}(0)\rangle=0,\tag{25}$$

provided that $q_1^\tau(t)^a_\nu$, as given in (24), can be interpreted as the (a,ν) component of a vector $|q_1^\tau(t)\rangle$ of $\mathscr{H}(I^\tau)$. We show this, and that $|q_1^\tau(t)\rangle$ is continuously differentiable at least in all cases (6): $q_1^\tau(t)^a_\nu(\kappa_1,\ldots,\kappa_\nu)$ is composed of the continuously differentiable elements of $G_0^{\tau+1}$ and of the continuously differentiable functions $g^{a,c}_j(t)$. Since the former are square integrable on $\mathscr{K}^{\times\nu}$, and since in both cases (6) the sum in (24) covers only a finite number of terms, the right-hand side of (24) certainly is again an element of $\mathscr{K}^{\times\nu}$ which is continuously differentiable with respect to t. If I and thus also I^τ are finite this is sufficient to make $|q_1^\tau(t)\rangle$ a continuously differentiable element of $\mathscr{H}(I^\tau)$. In the case (6*b*) we must show in addition that $\||q_1^\tau(t)\rangle\|$ is finite. In that case we have

$$\begin{aligned}
\||q_1^\tau(t)\rangle\|^2=\sum_{a\in Q}\sum_{\nu=1}^{\infty}\int d^3\kappa_1\ldots\int d^3\kappa_\nu\sum_{c\in Q}\sum_{c'\in Q}g^{*a,c}_{j_1}(t)g^{a,c'}_{j_1}(t)\\
\times\beta^*(^{b;\tau+\nu+1}_{0};t)^c_\nu(\kappa_1,\ldots,\kappa_\nu)\beta(^{b;\tau+\nu+1}_{0};t)^{c'}_\nu(\kappa_1,\ldots,\kappa_\nu).
\end{aligned}\tag{26}$$

Using the triangle relation separately for the sums over c and c' we get

$$\begin{aligned}
\||q_1^\tau(t)\rangle\|^2\leqslant\sum_{a\in Q}\sum_{\nu=0}^{\infty}\int d^3\kappa_1\ldots\int d^3\kappa\Bigg[\Bigg(\sum_{c\in Q}|g^{a,c}_{j_1}(t)|^2\Bigg)^{1/2}\\
\times\Bigg(\sum_{c\in Q}|\beta(^{b;\tau+\nu+1}_{0};t)^c_\nu(\kappa_1,\ldots,\kappa_\nu)|^2\Bigg)^{1/2}\Bigg]^2.
\end{aligned}\tag{27}$$

The last square is due to the fact that the sum over c' yields the same as the written sum over c. But now the integrations and the sum over ν act only on the second term and yield the square C^2 of the norm C of a norm-conserving vector of $\mathscr{H}(I)$. So we get

$$\| |q_1^\tau(t)\rangle \|^2 \leqslant \sum_{a\in Q} \sum_{c\in Q} |g_{j_1}^{a;c}(t)|^2 \times C^2 \tag{28}$$

and in the case (6*b*) this is finite.

Thus equation (25) is proved to be a well defined inhomogeneous, linear differential equation on $\mathscr{H}(I^\tau)$ with, at least in all cases (6), a self-adjoint generator H_{I^τ}. Therefore, under the initial condition (25) it has a unique solution (Kato 1966, p 486) which again is continuously differentiable in t.

By induction with respect to σ we find similarly that the elements of each generation G_σ exist and are continuously differentiable in t. Note that family G_σ^τ is the 'descendent' of family $G_0^{\tau+\sigma}$.

In the practically most important case of a finite I there exists a noteworthy 'progress from generation to generation'. With increasing τ the spaces $\mathscr{H}(I^\tau)$ contain states of fewer and fewer photons and finally become empty. The amplitudes of the last non-trivial founder family therefore describe the smallest number of photons and thus are the simplest to handle. The amplitudes in the first descendent $G_1^{\tau-1}$ describe one photon more than those of the parent family G_0^τ and thus are less simple; in general, the WW theory determining them also is of higher order in the sense of the GE hierarchy. The descendents of a given founder family thus become loaded with more and more photons and are thus less and less simple. But as soon as the descendents G_τ^0 of a G_0^τ have 'achieved' the Hilbert space $\mathscr{H}(I)$ and Hamiltonian H_I of the full (considered) WW theory, they remain without descendents and thus 'die out'. It is interesting that the considered theory is composed of elements of theories on Hilbert spaces $\mathscr{H}(I^\tau)$ which even can be orthogonal to $\mathscr{H}(I)$, namely if $I \cap I^\tau = \phi$. This can occur, eg in the example $I = (A, N), (B, N+1)$ of a two-element set I.

We have discussed the equations (17), (18) of condition (ii) only as far as needed for the proof of the composition law. The *actual* solution requires more work. We know from a number of rather representative WW theories that these solutions also can be 'composed' of known elements, namely the $M(a, b; \kappa)$, and of functions of t which depend only on the 'radial variables' $|\boldsymbol{k}_1|, \ldots, |\boldsymbol{k}_\nu|$ in the spaces $\mathscr{K}^{\times\nu}$, and are governed by corresponding 'radial equations'. We expect quite generally a sort of separation of the angle and polarization variables in $\mathscr{K}^{\times\nu}$ for any solution of (17). This certainly will ease the physical interpretation which therefore has been postponed. We note only that all results on the directions of photons described by solutions of (17) depend solely on the atomic transition elements $M(a, b; \kappa)$ and thus are independent of the directions of incident photons. For the constituents in G_0 this is clear because all elements of G_0 are amplitudes of *spontaneously* emitted photons. The constituents in later generations depend on the incident photons only through the *scalar* functions $g_j^{a,b}(t)$ which are quite insensitive to the latter's directions. For $\sigma \geqslant 1$ the elements of G_σ are amplitudes of photons emitted in resonance fluorescence processes after the absorption of σ photons and thus equipped with practically the same directivity characteristics as the spontaneous amplitudes of G_0. It is characteristic that there are no constituents which could be associated with 'stimulated' emission processes. Stimulated emission is a consequence of the process of composition, as we shall see.

We note finally that in a possible 'exact' theory the condition (i) was sufficient for the existence of all constituents which in that case simply were special solutions.

5. The proof of the composition law

The proof of our theorem is merely a technical verification but because of its complexity and the insight into the structure of WW theories it may provide it should not be omitted. It also reflects traces of the ideas and the labour needed to find the theorem at all.

We begin with part A and thus assume conditions (i), (ii) and (iii) to be satisfied.

We show first that the initial conditions (3) are satisfied by (16). The factor $\delta_{\nu,0}\delta_{\sigma,0}$ in (18) means that only the terms $\rho = m = n$ in (16) contribute to (16) at $t = 0$. The factor $\delta_{a,b}$ in (18) therefore further reduces the sum over $(b, m) \in I$ to the one term $(b, m) = (a, n)$. Thus (16) is reduced to the expression on the right-hand side of (13) if we again use the fact that the permutation sum yields, in this case, only a factor $n!$ because it appears under the sum (12). The left-hand side of (13) equals the required initial value.

Since for $t < \infty$ a unique, unitary time evolution operator $U_I(t)$ exists by condition (i), the states $|\alpha_I(t)\rangle$ evolving from the vectors of any base of $\mathscr{H}(I)$ again are a 'rotated' base of $\mathscr{H}(I)$. Therefore the state $|\alpha_I(t)\rangle$ evolving from any initial $|\chi_I\rangle$ at any time $t < \infty$ can be expanded with respect to this rotated base with time-independent expansion coefficients. These coefficients in essence equal our $\chi'^a_n(i_1, \ldots, i_n)$ because by an identical reformulation of (16), consisting of an additional, formal symmetrization process with respect to $i_1, \ldots, i_m$ under the sum over $i_1, \ldots, i_m$ we could achieve that (16) assumes precisely the form of an expansion with respect to the mentioned rotated base. Therefore it is sufficient to prove the theorem for any base vector of $\mathscr{H}(I)$ as initial state; the convergence of the sum over $(b, m) \in I$ and over $i_1, \ldots, i_m$ is then assured automatically. If we take as a base of $\mathscr{H}(I)$ the sum of the bases in $\mathscr{H}^a_n$ chosen in § 3 it certainly lies in the dense set $\mathscr{D}$ of vectors $|\chi_I\rangle \in \mathscr{H}(I)$ with only a finite number of non-vanishing expansion coefficients $\chi'^a_n(i_1, \ldots, i_n)$ when (a, n) varies over the whole domain I. $\mathscr{D}$ is also contained in the domain of the Hamilton operator H_I. Therefore it is sufficient to show that expressions (16) *solve* the equations (2) for any finite set of non-vanishing expansion coefficients $\chi'^a_n(i_1, \ldots, i_n)$.

But then all sums in (16) are finite and $\mathrm{i}(\mathrm{d}/\mathrm{d}t)\alpha(t)^a_n(\kappa_1, \ldots, \kappa_n)$ is obtained by term-by-term differentiation. We get

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)^a_n(\kappa_1, \ldots, \kappa_n) = L_0 + L_1 + L_2 + L_3 + L_4 \tag{29}$$

with L_0 denoting the terms from the differentiation of the exponentials in (16) and $L_1 \ldots L_4$ defined as the contribution of the first ... fourth term of the right-hand side of (17) when the t derivatives of the β's are replaced by (17), in accordance with condition (ii). Further, let R_1, R_2, R_3 denote the terms obtained from the first, second, third term of the right-hand side of (2) if (16) is inserted and the operator H_I is applied under these sums, according to condition (iii). We then have to verify the equation

$$L_0 + L_1 + L_2 + L_3 + L_4 = R_1 + R_2 + R_3. \tag{30}$$

We do this by separately verifying the equations $L_0 + L_1 = R_1, L_3 = R_3$, and $L_2 + L_4 = R_3$.

We first prove $L_0 + L_1 = R_1$. Let S_0, S_1 denote the summation operators

$$S_0 = \sum_{(b,m)\in I} \sum_{i_1,\ldots,i_m} \chi'^a_n(i, \ldots, i_m) \ldots \tag{31}$$

$$S_1 = \left(\frac{m!}{n!}\right)^{1/2} \sum_{\rho=0}^{\min(n,m)} \frac{1}{\rho![(n-\rho)!]^{1/2}} \mathop{\mathsf{P}}_{\eta_1,\ldots,\eta_n}^{\kappa_1,\ldots,\kappa_n} b_{i_1}(\eta_1)\,\mathrm{e}^{-\mathrm{i}\omega(\eta_1)t} \ldots b_{i_\rho}(\eta_\rho)\,\mathrm{e}^{-\mathrm{i}\omega(\eta)t} \ldots \tag{32}$$

so that

$$\alpha(t)^a_n(\kappa_1, \ldots, \kappa_n) = S_0 S_1 \beta(^{b;n}_{m-\rho}; i_{\rho+1}, \ldots, i_m; t)^a_{n-\rho}(\eta_{\rho+1}, \ldots, \eta_n). \tag{33}$$

Then

$$L_0 := S_0 S_1 (\omega(\eta_1) + \ldots + \omega(\eta_\rho))\beta \ldots, \tag{34}$$

$$L_1 := S_0 S_1 (\omega(\eta_{\rho+1}) + \ldots + \omega(\eta_n) + E^a)\beta \ldots, \tag{35}$$

with the same $\beta \ldots$ as in (33). Adding these terms and taking into account that the resulting sum $\omega(\eta_1) + \ldots + \omega(\eta_n)$ is the same for any permutation $\eta_1, \ldots, \eta_n$ of $\kappa_1, \ldots, \kappa_n$ we can take the factor $(\omega(\kappa_1) + \ldots + \omega(\kappa_n) + E^a)$ out of the permutation sum and out of $S_0 S_1$. But then

$$L_0 + L_1 = (\omega(\kappa_1) + \ldots + \omega(\kappa_n) + E^a)\alpha(t)^a_n(\kappa_1, \ldots, \kappa_n) =: R_1, \tag{36}$$

as stated.

Next we show: $L_3 = R_3$. We have, with the above symbols,

$$L_3 = S_0 S_1 \frac{1}{(n-\rho)^{1/2}} \sum_{(c,n-1)\in I} \sum_{\mu=1}^{n-\rho} M(c, a; \eta_\mu)$$

$$\times \beta(^{b;n-1}_{m-\rho}; i_{\rho+1}, \ldots, i_m; t)^c_{n-\rho-1}(\eta_{\rho+1}, \ldots, \eta_{\rho+\mu-1}, \eta_{\rho+\mu+1}, \ldots, \eta_n). \tag{37}$$

We note that each term of the sum over μ, since occurring under the permutation sum, yields the same as the term $\mu = n-\rho$. The resulting factor $n-\rho$ together with the factor $(n-\rho)^{-1/2}$ from (37) will be combined with the factor $((n-\rho)!)^{-1/2}$ of (32). Taking into account that the term $\rho = n$ of (32), if it occurs, is empty in (37) we can rewrite (37) in the form

$$L_3 = S_0 S_1' \sum_{(c,n-1)\in I} M(c, a; \eta_n)\beta(^{b;n-1}_{m-\rho}; i_{\rho+1}, \ldots, i_m; t)^c_{n-\rho-1}(\eta_{\rho+1}, \ldots, \eta_{n-1}) \tag{38}$$

with S_1' obtained from S_1 by the replacements $(n-\rho)^k \to (n-1-\rho)^k$ and $\min(n, m) \to \min(n-1, m)$. By definition of R_3 we have

$$R_3 = \sum_{(c,n-1)\in I} \frac{1}{n^{1/2}} \sum_{\mu=1}^{n} M(c, a; \kappa_\mu) S_0 \left(\frac{m!}{(n-1)!}\right)^{1/2} \sum_{\rho=0}^{\min(n-1,m)} \frac{1}{\rho![(n-1-\rho)!]^{1/2}}$$

$$\times \mathop{\mathsf{P}}_{\eta_1,\ldots,\eta_{n-1}}^{\kappa_1,\ldots,\kappa_{\mu-1},\kappa_{\mu+1},\ldots,\kappa_n} b_{i_1} \ldots b_{i\rho}\beta \ldots \tag{39}$$

with the same factors b_i and $\beta \ldots$ as contained in (38). Exchanging the sums over $(c, n-1)$ and the sum over μ with S_0, in accordance with condition (iii), and using the identity

$$\sum_{\mu=1}^{n} M(c, a; \kappa_\mu) \mathop{\mathsf{P}}_{\eta_1,\ldots,\eta_{n-1}}^{\kappa_1,\ldots,\kappa_{\mu-1},\kappa_{\mu+1},\ldots,\kappa_n} \ldots = \mathop{\mathsf{P}}_{\eta_1,\ldots,\eta_n}^{\kappa_1,\ldots,\kappa_n} M(c, a; \eta_n) \ldots \tag{39a}$$

we get $R_3 = L_3$ because also all number factors combine properly.

The most complicated part of the proof is the verification of the remaining $R_2 = L_2 + L_4$. R_2 is defined as

$$R_2 = (n+1)^{1/2} \sum_{(c,n+1)\in I} \int \mathrm{d}^3\kappa M^*(a, c; \kappa) S_0 \left(\frac{m!}{(n+1)!}\right)^{1/2} \sum_{\rho=0}^{\min(m,n+1)} \frac{1}{\rho![(n+1-\rho)!]^{1/2}}$$
$$\times \mathop{\mathsf{P}}_{\eta_1,\ldots,\eta_{n+1}}^{\kappa,\kappa_1,\ldots,\kappa_n} b_{i_1}(\eta_1)' \ldots b_{i_\rho}(\eta_\rho)' \beta(^{b;n+1}_{m-\rho}; i_{\rho+1}, \ldots, i_m; t)^c_{n+1-\rho}(\eta_{\rho+1}, \ldots, \eta_{n+1}) \tag{40}$$

with the primes indicating omitted exponential factors. We use the identity

$$\mathop{\mathsf{P}}_{x_1,\ldots,x_{n+1}}^{y,y_1,\ldots,y_n} f(x_1, \ldots, x_{n+1}) = \mathop{\mathsf{P}}_{x_1,\ldots,x_n}^{y_1,\ldots,y_n} \sum_{\mu=1}^{n+1} f(x_1, \ldots, x_{\mu-1}, y, x_{\mu+1}, \ldots, x_n) \tag{41}$$

and consider the contribution R_2' of the terms $\mu = 1, \ldots, \rho$. In these terms the integration in (40), after the permitted exchange with S_0, acts only on one factor b'_{i_μ} and produces, in accordance with (7), a factor $g^{a,c}_{i_\mu}(t)$. So we get

$$R_2' = S_0 \left(\frac{m!}{n!}\right)^{1/2} \sum_{\rho=0}^{\min(m,n+1)} \frac{1}{\rho![(n+1-\rho)!]^{1/2}} \mathop{\mathsf{P}}_{\eta_1,\ldots,\eta_n}^{\kappa_1,\ldots,\kappa_n} \sum_{(c,n+1)\in I}$$
$$\times \sum_{\mu=1}^{\rho} b_{i_1}(\eta_1)' \ldots b_{i_{\mu-1}}(\eta_{\mu-1})' g^{a,c}_{i_\mu}(t) b_{i_{\mu+1}}(\eta_\mu)' \ldots b_{i_\rho}(\eta_{\rho-1})'$$
$$\times \beta(^{b;n+1}_{m-\rho}; i_{\rho+1}, \ldots, i_m; t)^c_{n+1-\rho}(\eta_\rho, \ldots, \eta_n). \tag{42}$$

Since i_μ is only a summation index of the sum over $i_1, \ldots, i_m$ we may rename it, say to j_m. We also may rename $i_1, \ldots, i_{\mu-1}$ to $j_1, \ldots, j_{\mu-1}$ and $i_{\mu+1}, \ldots, i_m$ to $j_\mu, \ldots, j_{m-1}$. This procedure yields for any value of μ the same expression and so the sum over μ, since carried out under the sum over $i_1, \ldots, i_m$, actually means a factor ρ only, which can be combined with the factor $(\rho!)^{-1}$ in (42). Since (42) so is empty for $\rho = 0$ we may replace the summation index ρ by $\rho+1$ so that, with $j_1, \ldots, j_m$ again renamed to $i_1, \ldots, i_m$, we get

$$R_2' = S_0 S_1 \sum_{(c,n+1)\in I} g^{a,c}_{i_m}(t) \beta(^{b;n+1}_{m-1-\rho}; i_{\rho+1}, \ldots, i_{m-1}; t)^c_{n-\rho}(\eta_{\rho+1}, \ldots, \eta_n). \tag{43}$$

Note that the 'old' $i_{\rho+1}$ has been replaced by $j_\rho = i_\rho$ ('new') which by $\rho \to \rho+1$ assumes its old position. Note also that the upper limit of the sum over the new ρ is $\min(m, n+1)-1 = \min(m-1, n)$. We have replaced it by $\min(m, n)$ because the term $\rho = m$ of (43) is empty, by condition (ii). If now once more the indices $i_1, \ldots, i_m$ are renamed so that

$$i_1, \ldots, i_{m-\rho-1}, i_{m-\rho}, i_{m-\rho+1}, \ldots, i_m \to i_1, \ldots, i_{m-\rho-1}, i_{m-\rho+1}, \ldots, i_m, i_{m-\rho}$$

the new expression for (43) becomes identical with L_4, by the latter's definition.

So finally we have to show $L_2 = R_2''$ where R_2'' is the contribution of the terms $\mu = \rho+1, \ldots, n+1$ of (41) to R_2. For $\rho = n+1$ this contribution to R_2 is empty so the upper limit of the sum over ρ may be replaced by $\min(m, n)$. Further, the integration in

(40) now acts on one of the κ variables of the symmetric β's, so the sum over μ yields only a factor $n+1-\rho$. With the above definition of S_1 we thus get

$$R_2'' = S_0 S_1 (n+1-\rho)^{1/2} \sum_{(c,n+1)\in I} \int \mathrm{d}^3\kappa M^*(a,c;\kappa) \times \beta^{(b;n+1}_{m-\rho};i_{\rho+1},\ldots,i_m;t)^c_{n+1-\rho}(\kappa,\eta_{\rho+1},\ldots,\eta_n). \tag{44}$$

This is identical with L_2 and all statements of Part A are verified.

Part B. The statements referring to condition (i) are proven in GE, those referring to condition (ii) in § 4. In the cases (6) the condition (iii) is trivial because the critical sums contain only a finite number of terms.

The composition law of § 3 thus is proven in all details. With respect to validity, the theorem obviously can compete with results holding 'in any order of perturbation theory'. In view of this the proof is remarkably simple.

We illustrate the potential and the working mechanism of the composition law by applying it to some simple problems in the interaction of an atom with incident photons. The quest for non-trivial simplicity in a natural way will lead us to problems of optical coherence. So we restrict the considerations to the case (6*b*) where the photon number is not necessarily bounded. The statespace of the coupled system $A+R$ thus is a finite sum of Fock spaces $\mathscr{F}^a$

$$\mathscr{H}(I) = \bigoplus_{a\in Q} \mathscr{F}^a. \tag{45}$$

Optical coherence is usually discussed for the free radiation field R, ie on a single Fock space (mostly even only for a single mode, ie on a Hilbert space isomorphic to $L^2(\mathbb{R}^1)$). In our greater state space (45) we therefore in general can speak of the coherence of a state of R *only under the condition* that A is in some given state, eg some $u_a(x)$. If in a given situation we know that A is in some definite state, usually its ground state $u_A(x)$, in that situation we can speak of coherence in the usual way.

6. The relative weight of the factorization and the photon number uncertainty aspects of coherence

It is natural to assume that the initial states of A and R are 'prepared' independently of each other. This leads to an initial state of $A+R$ with components of the form

$$\chi_n^a(\kappa_1,\ldots,\kappa_n) = C^a \times \chi_n(\kappa_1,\ldots,\kappa_n) \tag{46}$$

where C^a and $\chi_n(\kappa_1,\ldots,\kappa_n)$ are independent of each other and subject to the normalization conditions

$$\sum_{a\in Q} |C^a|^2 = 1, \qquad \sum_{n=0}^{\infty} \int \mathrm{d}^3\kappa_1 \ldots \int \mathrm{d}^3\kappa_n |\chi_n(\kappa_1,\ldots,\kappa_n)|^2 = 1. \tag{47}$$

$|C^a|^2$ is the probability for A to be initially in the state $u_a(x)$ and the χ_n are the components of some given element $|\chi\rangle$ of the Fock space which is chosen as initial state of R. But (46) does not yet lead to remarkable simplifications. We achieve them, and this is probably

the most important result that can be derived from the composition law, *if and only if the* $\chi_n(\kappa_1, \ldots, \kappa_n)$ *for any* $n > 1$ *factor in the form*

$$\chi_n(\kappa_1, \ldots, \kappa_n) = c_n \times \varphi(\kappa_1) \times \ldots \times \varphi(\kappa_n). \tag{48}$$

The 'photon number distribution coefficients' c_n satisfy

$$\sum_{n=0}^{\infty} |c_n|^2 = 1 \tag{49}$$

and $\varphi(\kappa)$ is a given, normalized one-particle wavefunction in the momentum space. $|c_n|^2$ then is the probability that the initial state of R contains precisely n photons. It is most remarkable that the additional condition

$$c_n = \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^n}{(n!)^{1/2}} \tag{50}$$

with arbitrary $\alpha \in \mathbb{C}$ does not lead to further, physically relevant, simplifications. The radiation field R namely is initially in a 'modified vacuum state' (Friedrichs 1953) or in a 'fully coherent state' (Glauber 1963) with a mean of $\bar{N} := |\alpha|^2$ photons if and only if (50) *and* (48) hold. It is the 'factorization aspect' (48) of coherence which produces the desired simplicity; this aspect implies the assumption that all photons in the past, at $t < 0$, somehow have received the same preparation (Friedrichs 1953, Ernst 1969) and thus initially, at $t = 0$, are all 'in the same state', described in the momentum space by the wavepacket $\varphi(\boldsymbol{k}, \lambda)$ and in the position space by its Fourier transform $\tilde{\varphi}(\boldsymbol{x}, \lambda)$. If no interaction occurs later, the photons remain forever 'in the same state' but this 'state' changes in time in compliance with causality: $\varphi(\boldsymbol{k}, \lambda) \to \varphi(\boldsymbol{k}, \lambda) \times \mathrm{e}^{-\mathrm{i}t\omega(\boldsymbol{k})}$, $\tilde{\varphi}(\boldsymbol{x}, \lambda) \to \tilde{\varphi}(\boldsymbol{x}, \lambda; t)$. The position-space wavepacket $\tilde{\varphi}(\boldsymbol{x}, \lambda; t)$, the spatial Fourier transform of $\varphi(\boldsymbol{k}, \lambda)\mathrm{e}^{-\mathrm{i}\omega(\boldsymbol{k})t}$, moves and spreads, and thus transfers 'action', with the velocity of light. The desired simplicity mirrors the absence of particle correlations between the incident photons, and this is contained in (48). Our composition law thus once more reveals the distinguished role of the factoring many-photon amplitudes of form (48), as emphasized earlier (Ernst 1969).

Technically this simplicity is obtained as follows. We can, and shall, identify $\varphi(\kappa)$ with one of the base vectors $b_i(\kappa)$, say with $b_1(\kappa)$. The $\chi'^b_m(i_1, \ldots, i_m)$ occurring in (16) then assume the form

$$\chi'^b_m(i_1, \ldots, i_m) = C^b c_m \delta_{i_1 1} \times \ldots \times \delta_{i_m 1}. \tag{51}$$

All sums over $i_1, \ldots, i_m$ in (16) now can be carried out (this is the point) and only the few β functions $\beta(^{b;n}_{\sigma}; 1, \ldots, 1; t)^a_{\nu}(\kappa_1, \ldots, \kappa_{\nu})$ remain in (16). Since the index n and the 1's are superfluous, and for later convenience, we introduce functions $\gamma(^b_{\sigma}; t)^a_{\nu}(\kappa_1, \ldots, \kappa_{\nu})$ by putting

$$\beta(^{b;n}_{\sigma}; 1, \ldots, 1, t)^a_{\nu}(\kappa_1, \ldots, \kappa_{\nu}) = \gamma(^b_{\sigma}; t)^a_{\nu}(\kappa_1, \ldots, \kappa_{\nu}) \exp[-\mathrm{i}t(\omega(\kappa_1) + \ldots + \omega(\kappa_{\nu}) + E^a)]. \tag{52}$$

It is easily verified that the corresponding equations (17) are solved by (52) if the γ's

satisfy the equations

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\gamma(^{b}_{\sigma};t)^{a}_{\nu}(\kappa_1,\ldots,\kappa_\nu)$$

$$= (\nu+1)^{1/2}\sum_{c\in Q}\int \mathrm{d}^3\kappa M^*(a,c;\kappa)\exp[-\mathrm{i}t(\omega(\kappa)+E^c-E^a)]$$

$$\times\gamma(^{b}_{\sigma};t)^{c}_{\nu+1}(\kappa,\kappa_1,\ldots,\kappa_\nu)+\frac{1}{\nu^{1/2}}\sum_{c\in Q}\sum_{\mu=1}^{\nu}M(c,a;\kappa_\mu)$$

$$\times\exp[-\mathrm{i}t(\omega(\boldsymbol{k}_\mu)+E^a-E^c)]\,\gamma(^{b}_{\sigma};t)^{c}_{\nu-1}(\kappa_1,\ldots,\kappa_{\mu-1},\kappa_{\mu+1},\ldots,\kappa_\nu)$$

$$+\sum_{c\in Q}g_1^{a,c}(t)\exp[-\mathrm{i}t(E^c-E^a)]\,\gamma(^{b}_{\sigma-1};t)^{c}_{\nu}(\kappa_1,\ldots,\kappa_\nu) \tag{53}$$

under the initial conditions

$$\gamma(^{b}_{\sigma};0)^{a}_{\nu}(\kappa_1,\ldots,\kappa_\nu)=\delta_{b,a}\times\delta_{\sigma 0}\times\delta_{\nu 0}. \tag{54}$$

The composition law (16) then assumes the simple form

$$\alpha(t)^{a}_{n}(\kappa_1,\ldots,\kappa_n)=\exp[-\mathrm{i}t(\omega(\kappa_1)+\ldots+\omega(\kappa_n)+E^a)]\alpha'(t)^{a}_{n}(\kappa_1,\ldots,\kappa_n), \tag{55}$$

$\alpha'(t)^{a}_{n}(\kappa_1,\ldots,\kappa_n)$:

$$=\sum_{b\in Q}C^b\sum_{m=0}^{\infty}c_m\left(\frac{m!}{n!}\right)^{1/2}\sum_{\rho=0}^{\min(n,m)}\frac{1}{\rho![(n-\rho)!]^{1/2}}$$

$$\times\sum_{\kappa_1,\ldots,\kappa_n}^{\mathbf{P}}{}_{\eta_1,\ldots,\eta_n}\varphi(\eta_1)\times\ldots\times\varphi(\eta_\rho)\times\gamma(^{b}_{m-\rho};t)^{a}_{n-\rho}(\eta_{\rho+1},\ldots,\eta_n). \tag{56}$$

The exponential factor in (52) has been introduced to obtain the exponential factor in (55). $\alpha'(t)^{a}_{n}$ then is related to the Møller operator and thus, in general, possesses a limit for $t\to\infty$. For example, the probability $P^{a}_{n}(t)$ to find, at time t, A in the state $u_a(\boldsymbol{x})$ and R in a state of $n\geqslant 0$ photons, and each of them in that state which in the absence of interaction evolves from the initial state $\tilde{\varphi}(\boldsymbol{x},\lambda)$, is given by

$$P^{a}_{n}(t)=\left|\int\mathrm{d}^3\kappa_1\ldots\int\mathrm{d}^3\kappa_n\varphi^*(\kappa_1)\times\ldots\times\varphi^*(\kappa_n)\alpha'(t)^{a}_{n}(\kappa_1,\ldots,\kappa_n)\right|^2. \tag{57}$$

Introducing the projections

$$\Gamma(^{b}_{\sigma};t)^{a}_{\nu}:=\int\mathrm{d}^3\kappa_1\ldots\int\mathrm{d}^3\kappa_\nu\varphi^*(\kappa_1)\times\ldots\times\varphi^*(\kappa_\nu)\times\gamma(^{b}_{\sigma};t)^{a}_{\nu}(\kappa_1,\ldots,\kappa_\nu) \tag{58}$$

we obtain by straightforward calculations

$$P^{a}_{n}(t)=n!\left|\sum_{b\in Q}C^b\sum_{m=0}^{\infty}c_m(m!)^{1/2}\sum_{\rho=0}^{\min(m,n)}\frac{\Gamma(^{b}_{m-\rho};t)^{a}_{n-\rho}}{\rho![(n-\rho)!]^{1/2}}\right|^2. \tag{59}$$

The quantities (57)–(59) in general have a limit for $t\to\infty$.

A still more simple form of the composition law can be obtained only at the price of triviality. If c_m is specialized to (50) the factors $(m!)^{1/2}$ in (56) and (59) cancel, but this is physically not relevant. It also would have occurred in (16) if χ_n had been written in the form $\chi_n=c_n\chi_n''$ with c_n given by (50) and with any normalized, but not factoring χ_n''. Though in this case the 'photon number uncertainty coefficients' c_n are identical with those of a fully coherent state, the correlations in χ_n'' would lead to additional correlations

in the solution of the interaction problem which require the full apparatus of (16), (17). To emphasize once more the greater physical weight of the factorization aspect (48) of coherence we note that the c_m do not enter the physically decisive equations of motion (17) or (53). Like the C^b they occur in (16) or (56) only in the role of rather trivial coefficients of a quantum-mechanical linear composition of independent processes, whereas correlations in the incident beam, described by a non-factoring χ_n, directly enter the equations (17).

We have argued here mainly on the base of Friedrichs' (1953) discussion of modified vacuum states of the Fock space. For the reader more accustomed to the usual concept of coherence in one 'mode' (eg Glauber 1963, Paul 1963) the above remarks may appear strange. We therefore note that the state space of one mode is a Hilbert space isomorphic to an extremely small subspace of the Fock space, namely the set of states of form $\chi_n = c_n\phi(\kappa_1)\times \ldots \times \phi(\kappa_n)$ for all n, where the normalized $\phi(\kappa)$ is prescribed and fixed once and forever, and only the c_n can vary in time. Free motion and the related spread of action through the position space is not possible in such a theory. This requires a change $\chi(\kappa) \to \chi(\kappa)\,\mathrm{e}^{-\mathrm{i}t\omega(\boldsymbol{k})}$ whereas a given 'mode' changes only by a factor $\mathrm{e}^{-\mathrm{i}Et}$ with *constant* E. The decisive factorization aspect (48) of coherence in the Fock space thus is 'prescribed' by the one-mode assumption, and only the photon number uncertainty aspect is subject to analysis (eg Paul 1963, Brunner *et al* 1964). But to meet the requirements of causality at least with respect to the spread of action by photons we *must* consider a theory on a *complete* set of modes, or, as here and in GE, on a 'continuum of modes'. So as potential solutions we must admit all states of $\mathscr{H}(I)$, also those with complicated photon correlations, ie with photons 'distributed over many modes'. Our composition law owes its complexity to the fact that, unfortunately, the atom *does* use this freedom to emit 'into all modes'; we noted in § 4 and in a special case will see below that incident photons in essence do not actively hinder it from doing so.

7. An illustration of causality in Weisskopf–Wigner theories

To illustrate the above a little better we first must look at the function $g_1^{a,c}(t)$ which, in accordance with (7) and (4), can be written in the form

$$g_1^{a,c}(t) = \int \mathrm{d}^3k \left[\frac{e}{[2\omega(\boldsymbol{k})(2\pi)^3]^{1/2}} \left(\sum_{\lambda=1}^{2} \varphi(\boldsymbol{k},\lambda)\epsilon(\boldsymbol{k},\lambda)\,\mathrm{e}^{-\mathrm{i}t\omega(\boldsymbol{k})} \right) \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{x}} u_a^\dagger(\boldsymbol{x}) \alpha u_c(\boldsymbol{x}) \right]. \tag{60}$$

Let us assume that the incident wavepacket is similar to a Gaussian packet which moves in some direction and contains 'optical' photons of linewidth small compared with $|\bar{\boldsymbol{k}}|$, where $\bar{\boldsymbol{k}}$ is the mean photon momentum. Then $\varphi(\boldsymbol{k},\lambda)$ is 'concentrated' about $\bar{\boldsymbol{k}}$ and $1/(\omega(\boldsymbol{k}))^{1/2}$ can be considered constant in the region of the $\boldsymbol{k}$ space where $\varphi(\boldsymbol{k},\lambda)$ is different from zero. So it can be taken out of the integral (60). In the remaining integral we use the fact that the scalar product of two momentum-space wavefunctions equals the x-space scalar product of their Fourier transforms. So we get

$$g_1^{a,c}(t) = \frac{e}{[2\omega(\bar{\boldsymbol{k}})(2\pi)^3]^{1/2}} \int \mathrm{d}^3x \left[\left(\sum_{\lambda=1}^{2} \int \mathrm{d}^3k \varphi(\boldsymbol{k},\lambda)\epsilon(\boldsymbol{k},\lambda) \exp[-\mathrm{i}(\omega(\boldsymbol{k})t - \boldsymbol{k}\,.\,\boldsymbol{x})] \right) \times (u_a^\dagger(\boldsymbol{x})\alpha u_c(\boldsymbol{x})) \right]. \tag{61}$$

The second factor of the integrand is different from zero only in a region about the position $X = 0$ of the atom of dimensions of the Bohr radius a_0 which, under the above assumptions, satisfies $a_0 \ll |\bar{k}|^{-1}$. The first factor then for any t is nearly constant in that region and so it can be taken out of the x integral:

$$g_1^{a,c}(t) = \frac{e}{[2\omega(\bar{k})(2\pi)^3]^{1/2}} \phi(t, 0) \int d^3x u_a^\dagger(x) \alpha u_c(x). \tag{62}$$

The remaining integral exists and defines a constant current $J^{a,c}$ which depends only on atomic properties. The vector-valued function $\phi(t, 0)$ is defined as the value of the first factor of (61) at the position of the atom at time t. It is composed of the amplitudes of the incident wavepackets $\tilde{\varphi}(x, \lambda)$ at the position of the atom at time t. Therefore, $g_1^{a,c}(t)$ can be non-zero only as long as the incident wavepacket overlaps the atom. In particular, it vanishes for all t if the incident photo s do not 'hit' the atom at all.

A glance at (53) and (54) shows that the $\gamma(\ddot{\sigma}; t) \ldots$ for $\sigma \geqslant 1$ remain zero as long as A has not been reached by the incident wavepacket and that A radiates independently of the incident photons as soon as the overlapping has been ended. Within the limits of this example the composition law thus has disclosed that and how WW theories comply with the requirements of causality.

8. The single atom as square law detector and destroyer of coherence

Under the above realistic assumptions the overlap function $g_1^{a,c}(t)$ will be very small at any time, partly because of the factor e, but mainly due to the other conditions. If A, in addition, at $t = 0$ is in some excited state $u_B(x)$, ie $C^b = \delta_{b,B}$, the $\gamma(\ddot{\sigma} \ldots) \ldots$ for $\sigma \geqslant 1$ will be very small in comparison with the spontaneous amplitudes $\gamma(^B_0; t)^a_\nu(\kappa_1, \ldots, \kappa_\nu)$ which by definition are independent of the incident photons. Therefore, for a not too large number of incident photons the terms $\rho = m$ of (56) and (59) will be dominant. Neglect of the other terms leads to

$$\alpha'(t)^a_n(\kappa_1, \ldots, \kappa_n)$$

$$= \sum_{m=0}^{n} \frac{c_m}{[n!m!(n-m)!]^{1/2}} \mathop{\mathbf{P}}_{\eta_1, \ldots, \eta_n}^{\kappa_1, \ldots, \kappa_n} \varphi(\eta_1) \times \ldots \times \varphi(\eta_m)$$

$$\times \gamma(^B_0; t)^a_{n-m}(\eta_{m+1}, \ldots, \eta_n), \tag{63}$$

$$P_n^a(t) = n! \left| \sum_{m=0}^{n} c_m \frac{\Gamma(^B_0; t)^a_{n-m}}{[m!(n-m)!]^{1/2}} \right|^2. \tag{64}$$

Equation (63) emphasizes an other aspect of the composition law. In accordance with the Bose principle it 'composes' the state of $A + R$ even if there is no dynamical interaction between A and the incident photons, as here. This is so because incident photons and spontaneously emitted photons are quanta of the same field R and thus subject to the Bose principle. But even this 'composition' is not trivial; it still is the base of the concept of 'stimulated emission'.

Assume that $u_B(x)$ ($u_A(x)$) is the upper (lower) state of a 'two-level atom'. If in that case primarily *one* photon is emitted 'spontaneously', at $t \to \infty$ the amplitude $\gamma(^B_0; t)^A_1(\kappa)$ and its projection $\Gamma(^B_0; t)^A_1$ on the state of the incident photons will be dominant. The

sums in (63) and (64) then reduce to the terms $m = n-1$ and we get

$$\alpha'(\infty)_n^A(\kappa_1, \ldots, \kappa_n)$$

$$= \frac{c_{n-1}}{n^{1/2}} \sum_{\mu=1}^{n} \varphi(\kappa_1) \times \ldots \times \varphi(\kappa_{\mu-1}) \mathscr{S}(_0^B; \infty)_1^A(\kappa_\mu)\, \varphi(\kappa_{\mu+1}) \times \ldots \times \varphi(\kappa_n), \qquad (65)$$

$$P_{n+1}^A(\infty) = (n+1)|c_n \Gamma(_0^B; \infty)_1^A|^2, \qquad (P_0^A = 0). \qquad (66)$$

If the initial state contains precisely N photons, $c_n = \delta_{nN}$, (66) becomes the Fock space counterpart, $P_{N+1}^A(\infty) = (N+1)|\Gamma(_0^B; \infty)_1^A|^2$ of the historic factor $N+1$ of stimulated emission (Dirac 1927b) of first-order perturbation theory. Since $\mathscr{S}(_0^B; t)_1^A(\kappa)$ is not restricted to perturbation theory we actually have obtained a generalization of this theory. We want to emphasize here only that the 'stimulated emission factor' $n+1$ in (66) occurs independently of the c_n. Within the limits of this derivation (actually within a much wider frame) the atom does not react on the photon number uncertainty aspect of coherence; it has no 'sense' for it, does not feel it at all, and reacts to each 'n-photon component of the initial state' independently of the others. In other words, the atom behaves like a photon counter (square law detector), but not as a 'test body' for the measurement of electromagnetic field strengths.

It has been shown (Paul 1963) that an initially excited atom which can emit only one photon and is coupled to one mode so that it must emit its photon into that mode, in essence changes a fully coherent state of $\bar{N}$ photons to a fully coherent state of $\bar{N}+1$ photons in that mode. The situation is quite different in our case. We could speak of a fully coherent state of R at $t \to \infty$, when the atom is in its ground state, if expressions (63) and (65) could be written in the form (48) with a new $\varphi(\kappa)$ and a c_n of form (50) with a new α. But this is not possible, not even in a crude approximation. The point is that within very wide limits of validity, the approximations (63) and even (65) are very good, and that the γ's in (63) and (65) are independent of the incident photons. $\Gamma(_0^B; \infty)_1^A$, which in a self-consistent one-mode theory necessarily is of order 1 (cf Paul 1963), in our case is very small, but not zero under non-trivial circumstances. The full coherence of the initial state therefore is always destroyed to a finite, appreciable degree.

We emphasize that the above statements refer to a single atom. Many atoms, simultaneously coupled to 'all modes of R', can behave quite differently. More on these problems and other applications of the composition law will be given elsewhere.

Note added in proof. Our remarks on the limits $t \to \infty$ of the Møller operator and of (56), (59) are too optimistic. In fact, these limits exist in WW theories only in exceptional cases, eg if the atom is considered as a two level atom with sufficiently large energy difference $E^B - E^A$ and $I = \{(B, 0), (A, 1)\}$. Even a simple spin degeneracy of the levels makes these limits non-existent. This of course does not affect the existence of a unitary time evolution to any finite t. It also does not affect the composition law.

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